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Integration on time scales

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Abstract

In this paper we study the process of Riemann and Lebesgue integration on time scales. The relationship of the Riemann and Lebesgue integrals is considered and a criterion for Riemann integrability is established.

Keywords: Time scales; Delta and nabla derivatives; Delta and nabla integrals

1. Introduction

The concepts of the Riemann and Riemann–Stieltjes integrals on time scales were investigated earlier by Sailer in [21] where only the Darboux definition of the integral was considered. Below in Section 2 we give also the Riemann definition of the integral on time scales and prove the equivalence of the Darboux and Riemann definitions of the integral. In order to meet the requirements in applications (see [4]), we distinguish the concepts of delta and nabla integrals.

In Section 3 main properties of the Riemann integral are presented and in Section 4 two versions of the fundamental theorem of calculus are given.

The main part of this paper is Section 5. Here definitions of the Lebesgue delta and nabla measures and integrals on time scales are introduced. A comparison of the Lebesgue integral with the Riemann integral is given and an analogue of the classical Lebesgue criterion for Riemann integrability is proved. Note that the concept of the Lebesgue integral on time scales was briefly presented earlier in [4] and also independently in [7].

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Finally, in Appendix A, the terminology connected to time scales is stated and some used in Section 4 mean value theorems on time scales are given.

2. The Riemann delta and nabla integrals

Let $\mathbf{T}$ be a time scale, $a<b$ be points in $\mathbf{T}$, and $[a,b)$ be the half-closed bounded interval in $\mathbf{T}$. A *partition* of $[a,b)$ is any finite ordered subset $P=\{t_0,t_1,\ldots,t_n\}\subset[a,b]$, where $a=t_0<t_1<\cdots<t_n=b$. The number n depends on the particular partition, so we have $n=n(P)$. The intervals $[t_{i-1},t_i)$, $i=1,2,\ldots,n$, we call the subintervals of the partition P. Let f be a real-valued bounded function on $[a,b)$. Let us set $M=\sup\{f(t)\colon\ t\in[a,b)\}$, $m=\inf\{f(t)\colon\ t\in[a,b)\}$, $M_i=\sup\{f(t)\colon\ t\in[t_{i-1},t_i)\}$, $m_i=\inf\{f(t)\colon\ t\in[t_{i-1},t_i)\}$. The *upper Darboux Δ-sum* $U(f,P)$ and the *lower Darboux Δ-sum* $L(f,P)$ of f with respect to P are defined by

$$U(f,P)=\sum_{i=1}^{n}M_i(t_i-t_{i-1}),\qquad L(f,P)=\sum_{i=1}^{n}m_i(t_i-t_{i-1}).$$

It follows that

$$m(b-a)\leqslant L(f,P)\leqslant U(f,P)\leqslant M(b-a). \tag{2.1}$$

The *upper Darboux Δ-integral* $U(f)$ of f from a to b is defined by

$$U(f)=\inf\bigl\{U(f,P)\colon\ P \text{ is a partition of } [a,b)\bigr\}$$

and the *lower Darboux Δ-integral* $L(f)$ is

$$L(f)=\sup\bigl\{L(f,P)\colon\ P \text{ is a partition of } [a,b)\bigr\}.$$

In view of (2.1), $U(f)$ and $L(f)$ are finite real numbers. We will see in Theorem 2.4 that $L(f)\leqslant U(f)$.

Definition 2.1. We say that f is *Δ-integrable* (*delta integrable*) from a to b (or on $[a,b)$) provided $L(f)=U(f)$. In this case, we write $\int_a^b f(t)\,\Delta t$ for this common value. We call this integral the *Darboux Δ-integral*.

Riemann's definition of the integral is a little different (Definition 2.10), but we will show in Theorem 2.11 that the definitions are equivalent. For this reason, we will also call the integral defined above the *Riemann Δ-integral*.

If P and Q are two partitions of $[a,b)$ such that every point of P belongs to Q, i.e., $P\subset Q$, then we say that Q is a *refinement* of, or is *finer* than, P. The following lemma can be proved exactly in the same way as that in the case $\mathbf{T}=\mathbf{R}$ (see, for example, [20, Chapter VI]).

Lemma 2.2. *Let f be a bounded function on $[a,b)$. If P and Q are partitions of $[a,b)$ and Q is a refinement of P, then $L(f,P)\leqslant L(f,Q)\leqslant U(f,Q)\leqslant U(f,P)$, i.e., adding more points to a partition increases the lower sum and decreases the upper sum.*

Lemma 2.3. *If f is a bounded function on $[a,b)$, and if P and Q are any two partitions of $[a,b)$, then $L(f,P) \leqslant U(f,Q)$, i.e., every lower sum is less than or equal to every upper sum.*

Proof. The set $P \cup Q$ is also a partition of $[a,b)$. Since $P \subset P \cup Q$ and $Q \subset P \cup Q$, we can apply Lemma 2.2 to obtain $L(f,P) \leqslant L(f,P \cup Q) \leqslant U(f,P \cup Q) \leqslant U(f,Q)$. □

Obviously, Lemma 2.3 yields the following result.

Theorem 2.4. *If f is a bounded function on $[a,b)$, then $L(f) \leqslant U(f)$.*

It follows that $L(f,P) \leqslant L(f) \leqslant U(f) \leqslant U(f,Q)$ for all partitions P and Q of $[a,b)$. In particular, $L(f,P) \leqslant L(f) \leqslant U(f) \leqslant U(f,P)$ for all partitions P of $[a,b)$. Hence we get the following result:

Theorem 2.5. *If $L(f,P) = U(f,P)$ for some partition P of $[a,b)$, then the function f is Δ-integrable from a to b and $\int_a^b f(t)\,\Delta t = L(f,P) = U(f,P)$.*

The next theorem gives a "Cauchy criterion" for integrability and its proof is the same as that in the case $\mathbf{T} = \mathbf{R}$.

Theorem 2.6. *A bounded function f on $[a,b)$ is Δ-integrable if and only if for each $\varepsilon > 0$ there exists a partition P of $[a,b)$ such that $U(f,P) - L(f,P) < \varepsilon$.*

Lemma 2.7. *For every $\delta > 0$ there exists at least one partition P: $a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b)$ such that for each $i \in \{1,2,\ldots,n\}$ either $t_i - t_{i-1} \leqslant \delta$ or $t_i - t_{i-1} > \delta$ and $\rho(t_i) = t_{i-1}$, where ρ denotes the backward jump operator in $\mathbf{T}$.*

The proof can be found in [13,14,21].

Definition 2.8. For given $\delta > 0$ we denote by $\wp_\delta([a,b))$ or simply by $\wp_\delta$ the set of all partitions P: $a = t_0 < t_1 < \cdots < t_n = b$ that possess the property indicated in Lemma 2.7.

The following criterion for integrability can be proved as in the case $\mathbf{T} = \mathbf{R}$ (see [20, Chapter VI]).

Theorem 2.9. *A bounded function f on $[a,b)$ is Δ-integrable if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$P \in \wp_\delta \quad \text{implies} \quad U(f,P) - L(f,P) < \varepsilon \tag{2.2}$$

for all partitions P of $[a,b)$.

We now give Riemann's definition of integrability.

Definition 2.10. Let f be a bounded function on $[a, b)$, and let P: $a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a, b)$. In each interval $[t_{i-1}, t_i)$, where $1 \leqslant i \leqslant n$, choose an arbitrary point ξ_i and form the sum

$$S = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}). \tag{2.3}$$

We call S a *Riemann Δ-sum* of f corresponding to the partition P. We say that f is *Riemann Δ-integrable* from a to b (or on $[a, b)$) if there exists a number I with the following property. For each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S - I| < \varepsilon$ for every Riemann Δ-sum S of f corresponding to a partition $P \in \wp_\delta$ independent of the way in which we choose $\xi_i \in [t_{i-1}, t_i)$, $i = 1, 2, \ldots, n$. It is easily seen that such a number I is unique. The number I is the *Riemann Δ-integral* of f from a to b.

Theorem 2.11. *A bounded function f on $[a, b)$ is Riemann Δ-integrable if and only if it is* (*Darboux*) *Δ-integrable, in which case the values of the integrals are equal.*

Proof. Suppose first that f is (Darboux) Δ-integrable from a to b in the sense of Definition 2.1. Let $\varepsilon > 0$ and $\delta > 0$ be chosen so that (2.2) of Theorem 2.9 holds. We show that

$$\left| S - \int_a^b f(t)\,\Delta t \right| < \varepsilon \tag{2.4}$$

for every Riemann Δ-sum (2.3) associated with a partition $P \in \wp_\delta$. Clearly we have $L(f, P) \leqslant S \leqslant U(f, P)$ and so (2.4) follows from the inequalities

$$U(f, P) < L(f, P) + \varepsilon \leqslant L(f) + \varepsilon = \int_a^b f(t)\,\Delta t + \varepsilon,$$

$$L(f, P) > U(f, P) - \varepsilon \geqslant U(f) - \varepsilon = \int_a^b f(t)\,\Delta t - \varepsilon.$$

This proves (2.4); hence f is Riemann Δ-integrable and $I = \int_a^b f(t)\,\Delta t$.

Now suppose that f is Riemann Δ-integrable in the sense of Definition 2.10, and consider $\varepsilon > 0$. Let $\delta > 0$ and let I be as given in Definition 2.10. Select any partition P: $a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b)$ such that $P \in \wp_\delta$, and for each $i = 1, 2, \ldots, n$, choose ξ_i in $[t_{i-1}, t_i)$ so that $f(\xi_i) < m_i + \varepsilon$, where $m_i = \inf\{f(t)\colon t \in [t_{i-1}, t_i)\}$. The Riemann Δ-sum S for this choice of ξ_i's satisfies $S < L(f, P) + \varepsilon(b - a)$ as well as $|S - I| < \varepsilon$. It follows that $L(f) \geqslant L(f, P) > S - \varepsilon(b - a) > I - \varepsilon - \varepsilon(b - a)$. Since ε is arbitrary, we conclude that $L(f) \geqslant I$. A similar argument shows that $U(f) \leqslant I$. Since $L(f) \leqslant U(f)$, we see that $L(f) = U(f) = I$. This shows that f is (Darboux) Δ-integrable and that $\int_a^b f(t)\,\Delta t = I$. $\square$

In our definition of $\int_a^b f(t)\,\Delta t$, we assumed that $a < b$. We remove that restriction with the following definitions:

$$\int_a^a f(t)\,\Delta t = 0, \qquad \int_a^b f(t)\,\Delta t = -\int_b^a f(t)\,\Delta t, \quad a > b. \tag{2.5}$$

Theorem 2.12. *Assume that a and b are arbitrary points in $\mathbf{T}$. Every constant function $f(t) = c (t \in \mathbf{T})$ is Δ-integrable from a to b and*

$$\int_a^b c\,\Delta t = c(b-a). \tag{2.6}$$

Proof. Let $a < b$. Consider a partition $P\colon\ a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b)$. Obviously we have $U(f,P) = L(f,P) = c(b-a)$ and therefore Theorem 2.5 shows that f is Δ-integrable and (2.6) holds. Formula (2.6) for $a = b$ and $a > b$ follows by the definition (2.5). Note that every Riemann Δ-sum of f associated with P is also equal to $c(b-a)$. □

Theorem 2.13. *Let t be an arbitrary point in $\mathbf{T}$. Every function f defined on $\mathbf{T}$ is Δ-integrable from t to $\sigma(t)$ and*

$$\int_t^{\sigma(t)} f(s)\,\Delta s = \bigl[\sigma(t) - t\bigr] f(t). \tag{2.7}$$

Proof. If $\sigma(t) = t$, then (2.7) is obvious, because both sides of (2.7) are equal to zero in this case. Let now $\sigma(t) > t$. Then a single partition of $[t,\sigma(t))$ is $P\colon\ t = s_0 < s_1 = \sigma(t)$ and since $[s_0,s_1) = [t,\sigma(t)) = \{t\}$, we have $U(f,P) = f(t)[\sigma(t) - t] = L(f,P)$. Therefore, Theorem 2.5 shows that f is Δ-integrable from t to $\sigma(t)$ and (2.7) holds. Note that the Riemann Δ-sum of f associated with P is also equal to $f(t)[\sigma(t) - t]$. □

Theorem 2.14. *Assume $a, b \in \mathbf{T}$ and $a < b$. Then we have the following*:

(i) *If $\mathbf{T} = \mathbf{R}$, then a bounded function f on $[a,b)$ is Δ-integrable from a to b if and only if f is Riemann integrable on $[a,b)$ in the classical sense*; *in this case*

$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,dt,$$

where the integral on the right is the usual Riemann integral.

(ii) *If $\mathbf{T} = \mathbf{Z}$, then every function f defined on $\mathbf{Z}$ is Δ-integrable from a to b and*

$$\int_a^b f(t)\,\Delta t = \sum_{k=a}^{b-1} f(k). \tag{2.8}$$

Proof. Clearly, above given Definitions 2.1 and 2.10 of the Δ-integral coincide in case $\mathbf{T}=\mathbf{R}$ with the usual Darboux and Riemann definitions of the integral, respectively (see, e.g., [11,20]). Notice that the classical definitions of Darboux's and Riemann's integrals do not depend on whether the subintervals of the partition are taken closed, half-closed, or open. Moreover, if $\mathbf{T}=\mathbf{R}$, then $\wp_\delta([a,b))$ consists of all partitions of $[a,b)$ the norm (mesh) of which is less than or equal to δ. So part (i) of the theorem is valid. To prove part (ii), let $a<b$. Then $b=a+p$ for some positive integer p. Consider the partition P^* of $[a,b)$ defined by P^*: $a=t_0<t_1<\cdots<t_p=b$, where $t_0=a, t_1=a+1,\ldots,t_p=a+p$. P^* contains all points of $[a,b]$ and $[t_{i-1},t_i)=\{t_{i-1}\}$ for each $i\in\{1,2,\ldots,p\}$. Then

$$U(f,P^*)=\sum_{i=1}^{p}M_i(t_i-t_{i-1})=\sum_{i=1}^{p}f(t_{i-1}),$$
$$L(f,P^*)=\sum_{i=1}^{p}m_i(t_i-t_{i-1})=\sum_{i=1}^{p}f(t_{i-1}).$$

So $U(f,P^*)=L(f,P^*)=\sum_{i=1}^{p}f(t_{i-1})=\sum_{k=a}^{b-1}f(k)$ and Theorem 2.5 shows that f is Δ-integrable from a to b and (2.8) holds. □

Now we briefly describe the concept of ∇-*integral* (*nabla integral*) on time scales. Let P: $a=t_0<t_1<\cdots<t_n=b$ be a partition of $(a,b]$ and f be a real-valued bounded function on $(a,b]$. Let us set

$$M'=\sup\{f(t):\ t\in(a,b]\},\qquad m'=\inf\{f(t):\ t\in(a,b]\},$$
$$M_i'=\sup\{f(t):\ t\in(t_{i-1},t_i]\},\qquad m_i'=\inf\{f(t):\ t\in(t_{i-1},t_i]\}.$$

We call the sums

$$U'(f,P)=\sum_{i=1}^{n}M_i'(t_i-t_{i-1})\quad\text{and}\quad L'(f,P)=\sum_{i=1}^{n}m_i'(t_i-t_{i-1})$$

respectively as the *upper* and *lower Darboux* ∇-*sums* of f. It follows that

$$m'(b-a)\leqslant L'(f,P)\leqslant U'(f,P)\leqslant M'(b-a).$$

The numbers

$$U'(f)=\inf\{U'(f,P):\ P \text{ is a partition of } (a,b]\}$$

and

$$L'(f)=\sup\{L'(f,P):\ P \text{ is a partition of } (a,b]\}$$

are called the *upper* and *lower Darboux* ∇-*integrals* of f from a to b, respectively.

$U'(f)$ and $L'(f)$ are finite and the inequality $L'(f)\leqslant U'(f)$ holds.

We say that f is ∇-*integrable* (*nabla integrable*) from a to b (or on $(a,b]$) if $L'(f)=U'(f)$. In this case we write $\int_a^b f(t)\,\nabla t$ for this common value. We call this integral the *Darboux* ∇-*integral*.

A *Riemann* ∇*-sum* of f associated with the partition P is a sum of the form

$$S' = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

where $\xi_i \in (t_{i-1}, t_i]$ for $i = 1, 2, \ldots, n$. The function f is *Riemann* ∇*-integrable* from a to b (or on $(a, b]$) if there exists a number I' with the following property: For each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S' - I'| < \varepsilon$ for every Riemann ∇-sum S' of f associated with a partition $P \in \wp_\delta$, independent of the way in which we choose the points $\xi_i \in (t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$, where $\wp_\delta$ denotes as above the set of all partitions P of $(a, b]$ possessing the property indicated in Lemma 2.7 (note that the inequality $t_{i-1} < t_i$ with $\rho(t_i) = t_{i-1}$ is equivalent to $t_{i-1} < t_i$ with $\sigma(t_{i-1}) = t_i$). The number I' is the *Riemann* ∇*-integral* of f from a to b.

A bounded function f on $(a, b]$ is Riemann ∇-integrable if and only if it is (Darboux) ∇-integrable, in which case the values of the integrals agree.

In the case $\mathbf{T} = \mathbf{R}$, the Riemann ∇-integral, as in the case of the Δ-integral, coincides with the usual Riemann integral. In the case $\mathbf{T} = \mathbf{Z}$, we have $\int_a^b f(t)\,\nabla t = \sum_{k=a+1}^{b} f(k)$, $a < b$. Comparing this with (2.8) shows that the delta and nabla integrals are in general different.

Remark 2.15. Above in definition of the Δ-integral, we used as subintervals of a partition P: $a = t_0 < t_1 < \cdots < t_n = b$ the intervals $[t_{i-1}, t_i)$, $i = 1, 2, \ldots, n$. In [13,14] in definition of the Δ-integral the intervals $[t_{i-1}, \rho(t_i)]$, $i = 1, 2, \ldots, n$, were used instead of the intervals $[t_{i-1}, t_i)$, $i = 1, 2, \ldots, n$. It can be shown that the definitions are equivalent.

3. Basic properties of the Riemann integral

In this section we present some properties of the Riemann delta integral. Those properties hold also for the Riemann nabla integral accordingly.

Theorem 3.1. *Let f be Δ-integrable on $[a, b)$ and let M and m be its supremum and infimum on $[a, b)$, respectively. Let, further, $\varphi : \mathbf{R} \to \mathbf{R}$ be a function defined on $[m, M]$ such that there exists a positive constant B with $|\varphi(x) - \varphi(y)| \leqslant B|x - y|$ for all x and y in $[m, M]$* (*this condition is called as the Lipschitz condition*). *Then the composite function $h(t) = \varphi(f(t))$ is Δ-integrable on $[a, b)$.*

Proof. Consider an arbitrary $\varepsilon > 0$. By Theorem 2.6 there exists a partition P: $a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b)$ such that $U(f, P) - L(f, P) < \varepsilon/B$. Let M_i and m_i be the supremum and infimum of f on $[t_{i-1}, t_i)$, respectively, and let M_i^* and m_i^* be the corresponding numbers for h. By the condition on φ we have, for all s and τ in $[t_{i-1}, t_i)$,

$$\begin{aligned} h(s) - h(\tau) &\leqslant \bigl|h(s) - h(\tau)\bigr| = \bigl|\varphi\bigl(f(s)\bigr) - \varphi\bigl(f(\tau)\bigr)\bigr| \\ &\leqslant B\bigl|f(s) - f(\tau)\bigr| \leqslant B(M_i - m_i). \end{aligned}$$

Hence $M_i^* - m_i^* \leqslant B(M_i - m_i)$ because there exist two sequences (s_k) and (τ_k) of points in $[t_{i-1}, t_i)$ such that $h(s_k) \to M_i^*$ and $h(\tau_k) \to m_i^*$ as $k \to \infty$. Consequently,

$$U(h,P) - L(h,P) = \sum_{i=1}^{n} \bigl(M_i^* - m_i^*\bigr)(t_i - t_{i-1}) \leqslant B \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1})$$
$$= B\bigl[U(f,P) - L(f,P)\bigr] < B\frac{\varepsilon}{B} = \varepsilon$$

and h is Δ-integrable on $[a,b]$ by Theorem 2.6. □

The following theorem is more general than Theorem 3.1 and can be proved as in the case $\mathbf{T} = \mathbf{R}$.

Theorem 3.2. *Let f be Δ-integrable on $[a,b)$ and let M and m be its supremum and infimum on $[a,b)$, respectively. Let, further, $\varphi : \mathbf{R} \to \mathbf{R}$ be a continuous function on $[m, M]$. Then the composite function $h(t) = \varphi(f(t))$ is Δ-integrable on $[a,b)$.*

Corollary 3.3. *If f is Δ-integrable on $[a,b)$, then for an arbitrary positive number α the function $|f|^\alpha$ is Δ-integrable on $[a,b)$.*

Indeed, it is enough to consider the continuous function $\varphi(x) = |x|^\alpha$ and apply Theorem 3.2.

Theorem 3.4. *Let f be a bounded function that is Δ-integrable on $[a,b)$. Then f is Δ-integrable on every subinterval $[c,d)$ of $[a,b)$.*

Proof. Let $\varepsilon > 0$ and P be a partition of $[a,b)$ such that $U(f,P) - L(f,P) < \varepsilon$. Adding to P the points c and d we get a new partition P' of $[a,b)$. Then by Lemma 2.2 we also have $U(h,P') - L(h,P') < \varepsilon$. Now consider the partition P'' of $[c,d)$ consisting of all points of P' belonging to $[c,d]$. For upper and lower Δ-sums $\tilde{U}$ and $\tilde{L}$ of f on $[c,d)$ associated with this partition P'', we have $\tilde{U} - \tilde{L} \leqslant U(f,P') - L(f,P')$. So, $\tilde{U} - \tilde{L} < \varepsilon$ and by Theorem 2.6 the function f is Δ-integrable on $[c,d)$. □

Theorem 3.5. *Let f and g be Δ-integrable functions on $[a,b)$ and c be a real number. Then*

(i) *cf is Δ-integrable and $\int_a^b cf(t)\,\Delta t = c\int_a^b f(t)\,\Delta t$;*
(ii) *$f+g$ is Δ-integrable and $\int_a^b [f(t)+g(t)]\,\Delta t = \int_a^b f(t)\,\Delta t + \int_a^b g(t)\,\Delta t$.*

The proof is the same as that in the case $\mathbf{T} = \mathbf{R}$.

Theorem 3.6. *Let f and g be Δ-integrable functions on $[a,b)$. Then their product fg is Δ-integrable on $[a,b)$.*

Proof. First we prove that if f is Δ-integrable on $[a,b)$, then f^2 is Δ-integrable on $[a,b)$. Indeed, $f^2(t) = \varphi(f(t))$ with $\varphi(x) = x^2$, and φ satisfies the Lipschitz condition on any finite interval $[m, M]$. Therefore f^2 is integrable by Theorem 3.1. Now the desired result follows from the identity $4fg = (f+g)^2 - (f-g)^2$ by Theorem 3.5. $\square$

Theorem 3.7. *Let f be a function defined on $[a,b)$ and let $c \in \mathbf{T}$ with $a < c < b$. If f is Δ-integrable from a to c and from c to b, then f is Δ-integrable from a to b and*

$$\int_a^b f(t)\,\Delta t = \int_a^c f(t)\,\Delta t + \int_c^b f(t)\,\Delta t.$$

The proof is the same as that in the case $\mathbf{T} = \mathbf{R}$.

Theorem 3.8. *If f and g are Δ-integrable on $[a,b)$ and if $f(t) \leqslant g(t)$ for all $t \in [a,b)$, then $\int_a^b f(t)\,\Delta t \leqslant \int_a^b g(t)\,\Delta t$.*

Proof. By Theorem 3.5, $h = g - f$ is Δ-integrable on $[a,b)$. Since $h(t) \geqslant 0$ for all $t \in [a,b)$, it is clear that $L(h,P) \geqslant 0$ for all partitions P of $[a,b)$ and so $\int_a^b h(t)\,\Delta t = L(h) \geqslant 0$. Applying Theorem 3.5 again, we see that $\int_a^b g(t)\,\Delta t - \int_a^b f(t)\,\Delta t = \int_a^b h(t)\,\Delta t \geqslant 0$. $\square$

Theorem 3.9. *If f is Δ-integrable on $[a,b)$, then $|f|$ is Δ-integrable on $[a,b)$ and*

$$\left|\int_a^b f(t)\,\Delta t\right| \leqslant \int_a^b \left|f(t)\right|\Delta t. \tag{3.1}$$

Proof. This follows easily from Theorem 3.8 provided we know $|f|$ is Δ-integrable on $[a,b)$. In fact, $-|f| \leqslant f \leqslant |f|$ and so $-\int_a^b |f(t)|\,\Delta t \leqslant \int_a^b f(t)\,\Delta t \leqslant \int_a^b |f(t)|\,\Delta t$, which implies (3.1). We now show that $|f|$ is Δ-integrable. Consider the function $\varphi(x) = |x|$. This function satisfies a Lipschitz condition on any interval. Further, we have $|f(t)| = \varphi(f(t))$. Therefore $|f|$ is Δ-integrable by Theorem 3.1. $\square$

Corollary 3.10. *Let f and g be Δ-integrable on $[a,b)$. Then*

$$\left|\int_a^b f(t)g(t)\,\Delta t\right| \leqslant \int_a^b \left|f(t)g(t)\right|\Delta t \leqslant \Bigl(\sup_{t\in[a,b)} \left|f(t)\right|\Bigr) \int_a^b \left|g(t)\right|\Delta t.$$

Theorem 3.11. *Let (f_k) be a sequence of Δ-integrable functions on $[a,b)$ and suppose that $f_k \to f$ uniformly on $[a,b)$ for a function f defined on $[a,b)$. Then f is Δ-integrable from a to b and $\int_a^b f(t)\,\Delta t = \lim_{k\to\infty} \int_a^b f_k(t)\,\Delta t$.*

The proof is the same as that in the case $\mathbf{T} = \mathbf{R}$.

Here is a translation of Theorem 3.11 into theorem for series of functions.

Theorem 3.12. *Suppose that* $\sum_{k=1}^{\infty} g_k$ *is a series of* Δ*-integrable functions* g_k *on* $[a,b)$ *that converges uniformly to* g *on* $[a,b)$*. Then* g *is* Δ*-integrable and* $\int_a^b g(t)\,\Delta t = \sum_{k=1}^{\infty}\int_a^b g_k(t)\,\Delta t$.

4. Fundamental theorem of calculus

There are two versions of the fundamental theorem of calculus. Each says, roughly speaking, that differentiation and integration are inverse operations.

Let $[a,b]$ be a closed bounded interval in $\mathbf{T}$. A function $F:[a,b]\to\mathbf{R}$ is called a Δ-antiderivative of $f:[a,b)\to\mathbf{R}$ provided F is continuous on $[a,b]$ and Δ-differentiable on $[a,b)$, and $F^{\Delta}(t)=f(t)$ for all $t\in[a,b)$.

Theorem 4.1 (Fundamental theorem of calculus I). *Let* f *be* Δ*-integrable function on* $[a,b)$*. If* f *has a* Δ*-antiderivative* $F:[a,b]\to\mathbf{R}$*, then*

$$\int_a^b f(t)\,\Delta t = F(b)-F(a). \tag{4.1}$$

Proof. Let $\varepsilon>0$. By Theorem 2.6, there exists a partition P: $a=t_0<t_1<\cdots<t_n=b$ of $[a,b)$ such that

$$U(f,P)-L(f,P)<\varepsilon. \tag{4.2}$$

Applying Theorem A.2 (see Appendix A below) to $F:[t_{i-1},t_i]\to\mathbf{R}$ for each $i=1,2,\dots,n$, we obtain $\xi_i,\tau_i\in[t_{i-1},t_i)$ such that $(t_i-t_{i-1})f(\tau_i)\leqslant F(t_i)-F(t_{i-1})\leqslant(t_i-t_{i-1})f(\xi_i)$. Hence summing we have $\sum_{i=1}^{n}(t_i-t_{i-1})f(\tau_i)\leqslant F(b)-F(a)\leqslant\sum_{i=1}^{n}(t_i-t_{i-1})f(\xi_i)$. So the estimate

$$L(f,P)\leqslant F(b)-F(a)\leqslant U(f,P) \tag{4.3}$$

follows. Since we have $L(f,P)\leqslant\int_a^b f(t)\,\Delta t\leqslant U(f,P)$ for all partitions P of $[a,b)$, inequalities (4.2) and (4.3) imply that $|\int_a^b f(t)\,\Delta t-[F(b)-F(a)]|<\varepsilon$. Since ε is arbitrary, (4.1) holds. □

Theorem 4.2 (Integration by parts). *Let* u *and* v *be continuous functions on* $[a,b]$ *that are* Δ*-differentiable on* $[a,b)$*. If* u^{Δ} *and* v^{Δ} *are integrable from* a *to* b*, then*

$$\int_a^b u^{\Delta}(t)v(t)\,\Delta t+\int_a^b u\big(\sigma(t)\big)v^{\Delta}(t)\,\Delta t=u(b)v(b)-u(a)v(a). \tag{4.4}$$

Proof. Let $F=uv$; then $F^{\Delta}(t)=u^{\Delta}(t)v(t)+u(\sigma(t))v^{\Delta}(t)$ and F^{Δ} is Δ-integrable. Now Theorem 4.1 shows that $\int_a^b F^{\Delta}(t)\,\Delta t=F(b)-F(a)=u(b)v(b)-u(a)v(a)$ and so (4.4) holds. □

Theorem 4.3 (Fundamental theorem of calculus II). *Let f be a function which is Δ-integrable from a to b. For $t \in [a,b]$, let $F(t) = \int_a^t f(s)\,\Delta s$. Then F is continuous on $[a,b]$. Further, let $t_0 \in [a,b)$ and let f be arbitrary at t_0 if t_0 is right-scattered, and let f be continuous at t_0 if t_0 is right-dense. Then F is Δ-differentiable at t_0 and $F^\Delta(t_0) = f(t_0)$.*

Proof. Select $B > 0$ so that $|f(t)| \leqslant B$ for all $t \in [a,b)$. If $t, \tau \in [a,b]$ and $|t-\tau| < \varepsilon/B$ where $t < \tau$, say, then

$$\bigl|F(\tau) - F(t)\bigr| = \left|\int_t^\tau f(s)\,\Delta s\right| \leqslant \int_t^\tau \bigl|f(s)\bigr|\,\Delta s \leqslant \int_t^\tau B\,\Delta s = B(\tau - t) < \varepsilon.$$

This shows that F is (uniformly) continuous on $[a,b]$. Let $t_0 \in [a,b)$ be right-scattered. Then, since F is continuous, it is Δ-differentiable at t_0 and we have by Theorems 3.7 and 2.13,

$$\begin{aligned} F^\Delta(t_0) &= \frac{F(\sigma(t_0)) - F(t_0)}{\sigma(t_0) - t_0} = \frac{1}{\sigma(t_0) - t_0}\left[\int_a^{\sigma(t_0)} f(s)\,\Delta s - \int_a^{t_0} f(s)\,\Delta s\right] \\ &= \frac{1}{\sigma(t_0) - t_0}\int_{t_0}^{\sigma(t_0)} f(s)\,\Delta s = f(t_0), \end{aligned}$$

which is the desired result.

Suppose now that t_0 is right-dense and that f is continuous at t_0. In this case

$$\begin{aligned} F^\Delta(t_0) &= \lim_{t\to t_0}\frac{F(t) - F(t_0)}{t - t_0} = \lim_{t\to t_0}\frac{1}{t - t_0}\left[\int_a^{t} f(s)\,\Delta s - \int_a^{t_0} f(s)\,\Delta s\right] \\ &= \lim_{t\to t_0}\frac{1}{t - t_0}\int_{t_0}^{t} f(s)\,\Delta s. \end{aligned}$$

Let $\varepsilon > 0$. Since f is continuous at t_0, there exists $\delta > 0$ such that $s \in [a,b)$ and $|s - t_0| < \delta$ imply $|f(s) - f(t_0)| < \varepsilon$. Then

$$\begin{aligned} \left|\frac{1}{t - t_0}\int_{t_0}^{t} f(s)\,\Delta s - f(t_0)\right| &= \left|\frac{1}{t - t_0}\int_{t_0}^{t}\bigl[f(s) - f(t_0)\bigr]\,\Delta s\right| \\ &\leqslant \frac{1}{|t - t_0|}\left|\int_{t_0}^{t}\bigl|f(s) - f(t_0)\bigr|\,\Delta s\right| \leqslant \frac{\varepsilon}{|t - t_0|}\left|\int_{t_0}^{t}\Delta s\right| = \varepsilon \end{aligned}$$

for all $t \in [a,b]$ such that $|t - t_0| < \delta$ and $t \neq t_0$. Hence the desired result follows. □

A function $F : [a,b] \to \mathbf{R}$ is called a *Δ-preantiderivative* of $f : [a,b) \to \mathbf{R}$ provided F is continuous on $[a,b]$ and Δ-predifferentiable on $[a,b)$ with region of Δ-differentiation $D \subset [a,b)$ (see Appendix A below), and $F^\Delta(t) = f(t)$ for all $t \in D$.

Here is a generalization of Theorem 4.1.

Theorem 4.4. *Let f be a Δ-integrable function on $[a,b)$. If f has a Δ-preantiderivative $F:[a,b]\to\mathbf{R}$, then $\int_a^b f(t)\,\Delta t = F(b)-F(a)$.*

The proof is analogous to that of Theorem 4.1 and uses Theorem A.10 from Appendix A.

5. The Lebesgue delta and nabla integrals

For the main notions and facts from the general measure and integral theory we refer to [3,10,15,18].

Let $\mathbf{T}$ be a time scale, σ and ρ be the forward and backward jump functions, respectively, on $\mathbf{T}$.

Denote by $\Im_1$ the family (collection) of all left closed and right open intervals of $\mathbf{T}$ of the form $[a,b)=\{t\in\mathbf{T}\colon\ a\leqslant t<b\}$ with $a,b\in\mathbf{T}$ and $a\leqslant b$. The interval $[a,a)$ is understood as the empty set. $\Im_1$ is a semiring of subsets of $\mathbf{T}$. Let $m_1:\Im_1\to[0,\infty]$ be the set function on $\Im_1$ (whose values belong to the extended real half-line $[0,\infty]$) that assigns to each interval $[a,b)$ its length $b-a$: $m_1([a,b))=b-a$. Then m_1 is a countably additive measure on $\Im_1$. We denote by μ_Δ the Carathéodory extension of the set function m_1 associated with family $\Im_1$ (for the Carathéodory extension see [3]) and call μ_Δ the *Lebesgue Δ-measure* on $\mathbf{T}$.

Let us briefly describe the Carathéodory extension μ_Δ of m_1. First, using the pair $(\Im_1,m_1)$, we generate an outer measure m_1^* on the family of all subsets of $\mathbf{T}$ as follows. Let E be any subset of $\mathbf{T}$. If there exists at least one finite or countable system of intervals $V_j\in\Im_1$ $(j=1,2,\ldots)$ such that $E\subset\bigcup_j V_j$, then we put $m_1^*(E)=\inf\sum_j m_1(V_j)$, where the infimum is taken over all coverings of E by a finite or countable system of intervals $V_j\in\Im_1$. If there is no such covering of E, then we put $m_1^*(E)=\infty$. Second, we define the family $\mathbf{M}(m_1^*)$ of all m_1^*-measurable subsets of $\mathbf{T}$. A subset A of $\mathbf{T}$ is said to be m_1^*-measurable (or Δ-measurable) if $m_1^*(E)=m_1^*(E\cap A)+m_1^*(E\cap A^c)$ holds for all $E\subset\mathbf{T}$, where A^c denotes the complement of A: $A^c=\mathbf{T}-A$. The family $\mathbf{M}(m_1^*)$ of all m_1^*-measurable subsets of $\mathbf{T}$ is a σ-algebra. Third, we take the restriction of m_1^* to $\mathbf{M}(m_1^*)$, which we denote by μ_Δ. This μ_Δ (the *Lebesgue Δ-measure*) is a countably additive measure on $\mathbf{M}(m_1^*)$.

All intervals of family $\Im_1$ including the empty set are Δ-measurable. From the definition it is obvious that the $\mathbf{T}$ is also Δ-measurable. Suppose that $\mathbf{T}$ has a finite maximum τ_0. Obviously, the set $X=\mathbf{T}-\{\tau_0\}$ can be represented as a finite or countable union of intervals of the family $\Im_1$ and, therefore is Δ-measurable. Consequently, the single-point set $\{\tau_0\}=\mathbf{T}-X$ is Δ-measurable as the difference of two measurable sets $\mathbf{T}$ and X. Evidently, the single-point set $\{\tau_0\}$ does not have a finite or countable covering by intervals of $\Im_1$. Therefore, the single-point set $\{\tau_0\}$ and also any Δ-measurable subset of $\mathbf{T}$ containing τ_0 have Δ-measure infinity.

Remark 5.1. The fact that the Δ-measure of the max of the bounded above time scale is equal to infinity is a consequence of definition of the Δ-measure, given above. A source of this consequence is the assumption that if a subset E of the time scale is not coverable (by finite or countable systems of intervals from $\Im_1$) then its outer measure is assumed to be infinity. This assumption guarantees the needed monotonicity of an outer measure. We would also act differently in definition of outer measure not disturbing its monotonicity as follows. Let E be a noncoverable subset of $\mathbf{T}$ and let F be the maximal coverable subset of E. Then we define the outer measure of E to be the outer measure of F defined as above plus some fixed (independent of E) number c, where c belongs to the extended half-line $[0,\infty]$. In this case the outer measure (and hence the Δ-measure) of the max of $\mathbf{T}$ will be equal to c because the single-point set consisting of the max of $\mathbf{T}$ has not any coverable subset, that is, its coverable subset is only the empty set, and on the other hand the outer measure of the empty set is zero. Our previous definition of the outer measure corresponds to the value $c=\infty$.

Theorem 5.2. *For each t_0 in $\mathbf{T}-\{\max\mathbf{T}\}$ the single-point set $\{t_0\}$ is Δ-measurable and its Δ-measure is given by*

$$\mu_\Delta\big(\{t_0\}\big)=\sigma(t_0)-t_0. \tag{5.1}$$

Proof. If t_0 is right-scattered, then $\{t_0\}=[t_0,\sigma(t_0))\in\Im_1$. Therefore, $\{t_0\}$ is Δ-measurable and we have $\mu_\Delta(\{t_0\})=m_1([t_0,\sigma(t_0)))=\sigma(t_0)-t_0$, which is the desired result. Further consider the case when t_0 is right-dense. In this case there exists a decreasing sequence (t_k) of points of $\mathbf{T}$ such that $t_k>t_0$ and $t_k\to t_0$. Then $[t_0,t_1)\supset[t_0,t_2)\supset\cdots$ and $\{t_0\}=\bigcap_{k=1}^{\infty}[t_0,t_k)$. Hence $\{t_0\}$ is Δ-measurable as a countable intersection of Δ-measurable sets and by the continuity property of the countably additive measure μ_Δ, we have $\mu_\Delta(\{t_0\})=\lim_{k\to\infty}\mu_\Delta([t_0,t_k))=\lim_{k\to\infty}(t_k-t_0)=0$, and so (5.1) holds in this case as well. $\square$

Since each single-point subset of $\mathbf{T}$ is Δ-measurable and since every kind of interval can be obtained from an interval of the form $[a,b)$ by adding or subtracting the end points a and b, each interval of $\mathbf{T}$ is Δ-measurable. The following theorem gives formulas for evaluating the Δ-measure of any interval of $\mathbf{T}$.

Theorem 5.3. *If $a,b\in\mathbf{T}$ and $a\leqslant b$, then*

$$\mu_\Delta\big([a,b)\big)=b-a,\qquad \mu_\Delta\big((a,b)\big)=b-\sigma(a). \tag{5.2}$$

If $a,b\in\mathbf{T}-\{\max\mathbf{T}\}$ and $a\leqslant b$, then

$$\mu_\Delta\big((a,b]\big)=\sigma(b)-\sigma(a),\qquad \mu_\Delta\big([a,b]\big)=\sigma(b)-a. \tag{5.3}$$

Proof. The first formula of (5.2) is obvious, since $\mu_\Delta([a,b))=m_1([a,b))=b-a$. To prove the second one, we write $[a,b)=\{a\}\cup(a,b)$. Hence by the additivity of μ_Δ, we have $\mu_\Delta([a,b))=\mu_\Delta(\{a\})+\mu_\Delta((a,b))$. This gives, using Theorem 5.2, $b-a=\sigma(a)-a+\mu_\Delta((a,b))$ and, therefore, the second formula of (5.2) holds. The formulas of (5.3) can be proved in a similar way. $\square$

Now we define the concept of the Lebesgue ∇-measure on $\mathbf{T}$. Let $\Im_2$ denote the family of all right closed and left open intervals of $\mathbf{T}$ of the form $(a,b] = \{t \in \mathbf{T}\colon\ a < t \leqslant b\}$ with $a, b \in \mathbf{T}$ and $a \leqslant b$. The interval $(a,a]$ is understood as the empty set. $\Im_2$ is a semiring of subsets of $\mathbf{T}$. Further, let $m_2 : \Im_2 \to [0,\infty]$ be the set function on $\Im_2$ assigned to each interval $(a,b]$ its length $b-a$: $m_2((a,b]) = b-a$. Then m_2 is a countably additive measure on $\Im_2$. We denote by μ_∇ the Carathéodory extension of the set function m_2 associated with the family $\Im_2$ and call μ_∇ the *Lebesgue ∇-measure* on $\mathbf{T}$.

The following two theorems can be proved analogously to Theorems 5.2 and 5.3.

Theorem 5.4. *For each t_0 in $\mathbf{T} - \{\min \mathbf{T}\}$ the ∇-measure of the single-point set $\{t_0\}$ is given by $\mu_\nabla(\{t_0\}) = t_0 - \rho(t_0)$.*

Theorem 5.5. *If $a, b \in \mathbf{T}$ and $a \leqslant b$, then $\mu_\nabla((a,b]) = b - a$, $\mu_\nabla((a,b)) = \rho(b) - a$. If $a, b \in \mathbf{T} - \{\min \mathbf{T}\}$ and $a \leqslant b$, then $\mu_\nabla([a,b)) = \rho(b) - \rho(a)$, $\mu_\nabla([a,b]) = b - \rho(a)$.*

Remark 5.6. In the case $\mathbf{T} = \mathbf{R}$ both measures μ_Δ and μ_∇ coincide with the usual Lebesgue measure on $\mathbf{R}$. In the case $\mathbf{T} = \mathbf{Z}$ we also have $\mu_\Delta = \mu_\nabla$, and for any subset $E \subset \mathbf{Z}$, $\mu_\Delta(E) = \mu_\nabla(E)$ coincide with the number of points of the set E.

The Lebesgue integrals associated with the measures μ_Δ and μ_∇ on $\mathbf{T}$ we call the *Lebesgue Δ-integral* and the *Lebesgue ∇-integral* on $\mathbf{T}$, respectively. For a (measurable) set $E \subset \mathbf{T}$ and a (measurable) function $f : E \to \mathbf{R}$, the corresponding integrals of f over E we denote by $\int_E f(t)\,\Delta t$ and $\int_E f(t)\,\nabla t$, respectively.

So all theorems of the general Lebesgue integration theory, including the Lebesgue dominated convergence theorem, will be held for the Lebesgue delta and nabla integrals on $\mathbf{T}$.

Here is a comparison of the Lebesgue Δ-integral with the Riemann Δ-integral.

Theorem 5.7. *Let $[a,b)$ be a half-closed bounded interval in $\mathbf{T}$ and let f be a bounded real-valued function on $[a,b)$. If f is Riemann Δ-integrable from a to b, then f is Lebesgue Δ-integrable on $[a,b)$, and $R\int_a^b f(t)\,\Delta t = L\int_{[a,b)} f(t)\,\Delta t$, where R and L indicate the Riemann and Lebesgue integrals, respectively.*

Proof. Suppose that f is Riemann Δ-integrable from a to b. Then, by Theorem 2.9, for each positive integer k we can choose $\delta_k > 0$ ($\delta_k \to 0$ as $k \to \infty$) and a partition $P_k\colon\ a = t_0^{(k)} < t_1^{(k)} < \cdots < t_{n(k)}^{(k)} = b$ of $[a,b)$ such that $P_k \in \wp_{\delta_k}$ and $U(f,P_k) - L(f,P_k) < 1/k$. Hence

$$\lim_{k\to\infty} L(f,P_k) = \lim_{k\to\infty} U(f,P_k) = R\int_a^b f(t)\,\Delta t. \tag{5.4}$$

By replacing the partitions P_k with finer partitions if necessary, we can assume that for each k the partition P_{k+1} is a refinement of the partition P_k. Let us set

$$m_i^{(k)} = \inf\bigl\{f(t)\colon\ t \in \bigl[t_{i-1}^{(k)}, t_i^{(k)}\bigr)\bigr\}, \qquad M_i^{(k)} = \sup\bigl\{f(t)\colon\ t \in \bigl[t_{i-1}^{(k)}, t_i^{(k)}\bigr)\bigr\} \tag{5.5}$$

for $i = 1, 2, \ldots, n(k)$ and define sequences (φ_k) and (Φ_k) of functions on $[a, b)$ by letting

$$\varphi_k(t) = m_i^{(k)} \quad \text{and} \quad \Phi_k(t) = M_i^{(k)}, \quad \text{for } t \in \big[t_{i-1}^{(k)}, t_i^{(k)}\big),\ i = 1, 2, \ldots, n(k). \tag{5.6}$$

Then (φ_k) is a nondecreasing and (Φ_k) is a nonincreasing sequence of simple Δ-measurable functions. For each k we have

$$\varphi_k \leqslant \varphi_{k+1}, \qquad \Phi_k \geqslant \Phi_{k+1}, \tag{5.7}$$

$$\varphi_k \leqslant f \leqslant \Phi_k, \tag{5.8}$$

$$L \int\limits_{[a,b)} \varphi_k(t)\, \Delta t = L(f, P_k), \qquad L \int\limits_{[a,b)} \Phi_k(t)\, \Delta t = U(f, P_k). \tag{5.9}$$

Since f is bounded, the sequences (φ_k) and (Φ_k) are bounded by (5.5) and (5.6). Taking into account, in addition, the monotonicity (5.7) we conclude that there exist the limit functions

$$\varphi(t) = \lim_{k\to\infty} \varphi_k(t), \quad \Phi(t) = \lim_{k\to\infty} \Phi_k(t), \quad t \in [a, b). \tag{5.10}$$

It follows from (5.8) that

$$\varphi(t) \leqslant f(t) \leqslant \Phi(t), \quad t \in [a, b). \tag{5.11}$$

The functions φ and Φ are Δ-measurable as limits (5.10) of the Δ-measurable functions φ_k and Φ_k, respectively. Lebesgue's dominated convergence theorem implies that φ and Φ are Lebesgue Δ-integrable on $[a, b)$ and

$$\lim_{k\to\infty} L \int\limits_{[a,b)} \varphi_k(t)\, \Delta t = L \int\limits_{[a,b)} \varphi(t)\, \Delta t \quad \text{and}$$

$$\lim_{k\to\infty} L \int\limits_{[a,b)} \Phi_k(t)\, \Delta t = L \int\limits_{[a,b)} \Phi(t)\, \Delta t. \tag{5.12}$$

Therefore passing on to the limit in (5.9) as $k \to \infty$ and taking into account (5.4), we get

$$L \int\limits_{[a,b)} \varphi(t)\, \Delta t = L \int\limits_{[a,b)} \Phi(t)\, \Delta t = R \int\limits_a^b f(t)\, \Delta t. \tag{5.13}$$

From (5.13) we have

$$L \int\limits_{[a,b)} \big[\Phi(t) - \varphi(t)\big]\, \Delta t = 0. \tag{5.14}$$

Since in addition $\Phi(t) - \varphi(t) \geqslant 0$ for all t in $[a, b)$, (5.14) implies

$$\varphi(t) = \Phi(t) \quad \text{for } \Delta\text{-almost every } t \text{ in } [a, b). \tag{5.15}$$

So (5.15) and (5.11) show that $f(t) = \varphi(t)$ Δ-almost everywhere on $[a, b)$. It follows that f is Lebesgue Δ-integrable, and by (5.13), the Lebesgue Δ-integral of f over $[a, b)$ coincides with the Riemann Δ-integral of f from a to b. $\square$

The next theorem gives a "Lebesgue criterion" for Riemann Δ-integrability.

Theorem 5.8. *Let f be a bounded function defined on the half-closed bounded interval $[a,b)$ of $\mathbf{T}$. Then f is Riemann Δ-integrable from a to b if and only if the set of all right-dense points of $[a,b)$ at which f is discontinuous is a set of Δ-measure zero.*

Proof. Suppose that f is Riemann Δ-integrable from a to b. For each positive integer k let a partition P_k of $[a,b)$ and the functions φ_k, Φ_k, φ, and Φ be defined as in the proof of Theorem 5.7. Let us set

$$\Lambda = \bigcup_{k=1}^{\infty} P_k, \qquad \Lambda_{\mathrm{rd}} = \bigl\{t \in [a,b):\ t \in \Lambda \text{ and } t \text{ is right-dense}\bigr\}, \tag{5.16}$$

$$G = \bigl\{t \in [a,b):\ f \text{ is discontinuous at } t\bigr\}, \qquad G_{\mathrm{rd}} = \bigl\{t \in G:\ t \text{ is right-dense}\bigr\}, \tag{5.17}$$

$$A = \bigl\{t \in [a,b):\ \varphi(t) \neq \Phi(t)\bigr\}. \tag{5.18}$$

Assume t is a point in $[a,b)$ such that

$$\varphi(t) = f(t) = \Phi(t) \tag{5.19}$$

and that t is not in Λ. Then f is continuous at t; for otherwise there would exist $\varepsilon > 0$ and a sequence (t_j), $\lim t_j = t$, such that $|f(t_j) - f(t)| > \varepsilon$ for each j. But then $\Phi(t) \geqslant \varphi(t) + \varepsilon$ which contradicts the assumption (5.19).

Note that for each right-scattered point t of $[a,b)$ the equality (5.19) holds. Indeed, it is easy to see that all right-scattered points of $[a,b)$ belong to Λ (that is, they are partition points). Therefore for each right-scattered point t of $[a,b)$ and all sufficiently large k, we have $\varphi_k(t) = \Phi_k(t) = f(t)$, and hence $\varphi(t) = \Phi(t)$.

Consequently,

$$G_{\mathrm{rd}} \subset (A \cup \Lambda_{\mathrm{rd}}). \tag{5.20}$$

Further, $\mu_\Delta(A) = 0$ by the proof of Theorem 5.7 (see (5.15)) and $\mu_\Delta(\Lambda_{\mathrm{rd}}) = 0$ by the fact that Λ, and therefore Λ_{rd}, is at most countable, and that each right-dense point has Δ-measure zero. So from (5.20) we get that $\mu_\Delta(G_{\mathrm{rd}}) = 0$ and the first part of the theorem is proved.

Conversely, suppose that the set of all right-dense points of $[a,b)$ at which f is discontinuous is of Δ-measure zero: $\mu_\Delta(G_{\mathrm{rd}}) = 0$. For each positive integer k we choose $\delta_k > 0$ ($\delta_k \to 0$ as $k \to \infty$)and a partition P_k: $a = t_0^{(k)} < t_1^{(k)} < \cdots < t_{n(k)}^{(k)} = b$ of $[a,b)$ such that $P_k \in \wp_{\delta_k}$ and that P_{k+1} is a refinement of P_k. Further, let φ_k, Φ_k, φ, and Φ be defined as before. So for each k we have $\varphi_k \leqslant \varphi_{k+1}$, $\Phi_k \geqslant \Phi_{k+1}$, $\varphi_k \leqslant f \leqslant \Phi_k$ and $\varphi(t) = \lim_{k\to\infty} \varphi_k(t)$, $\Phi(t) = \lim_{k\to\infty} \Phi_k(t)$, $\varphi(t) \leqslant f(t) \leqslant \Phi(t)$, $t \in [a,b)$.

Now suppose that $t \in [a,b)$ is a right-dense point and that f is continuous at t. Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\sup f - \inf f < \varepsilon$, where the supremum and infimum are taken over $(t-\delta, t+\delta) = \{s \in [a,b):\ t-\delta < s < t+\delta\}$. For all k sufficiently large, a subinterval of P_k containing t will be lain in $(t-\delta, t+\delta)$, and so $\Phi_k(t) - \varphi_k(t) < \varepsilon$. But ε is arbitrary so $\varphi(t) = \Phi(t)$. Next, at each right-scattered point t of $[a,b)$ we have, as above in the first part of the proof, $\varphi(t) = \Phi(t)$.

So we have shown that $\varphi(t) = \Phi(t)$ provided either t is right-dense and f is continuous at t or t is right-scattered. Therefore we have the inclusion $A \subset G_{\mathrm{rd}}$, where A and G_{rd} are defined by (5.18) and (5.17), respectively. Hence, since $\mu_\Delta(G_{\mathrm{rd}}) = 0$ by our assumption, we get that $\mu_\Delta(A) = 0$. Consequently, $\varphi(t) = \Phi(t)$ Δ-almost everywhere on $[a, b)$. This implies that $L\int_{[a,b)} \varphi(t)\,\Delta t = L\int_{[a,b)} \Phi(t)\,\Delta t$ and, therefore, taking into account (5.12) and (5.9), we get $\lim_{k\to\infty} L(f, P_k) = \lim_{k\to\infty} U(f, P_k)$. This shows, by Theorem 2.6, that f is Riemann Δ-integrable on $[a, b)$. □

By Theorem 5.2 every right-dense point in $\mathbf{T}$ has the Δ-measure zero and hence, by countably additivity of the Δ-measure, every finite or countable set of right-dense points has Δ-measure zero. Consequently, applying Theorem 5.8 we get that every bounded function $f : [a, b) \to \mathbf{R}$ having only finitely or countably many right-dense discontinuity points in $[a, b)$ is Riemann Δ-integrable from a to b. In particular, every bounded continuous function on $[a, b)$ is Riemann Δ-integrable.

A function $f : \mathbf{T} \to \mathbf{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbf{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbf{T}$. Obviously, every continuous function is rd-continuous. The jump operator σ, in general, is not continuous but it is rd-continuous.

A function $f : \mathbf{T} \to \mathbf{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points in $\mathbf{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbf{T}$. We see that every rd-continuous and also every monotonic function is regulated.

It is no difficult to see that every regulated function on a compact interval is bounded (see [9, Theorem 1.65]). We now show that every regulated function on a compact interval may have at most countably many discontinuity points in this interval. It is sufficient to show that for each positive integer n the set

$$G_n = \left\{ t \in [a, b]\colon \left| f(t + 0) - f(t - 0) \right| \geqslant \frac{1}{n} \right\}$$

is finite. Then the set of all discontinuity points of f in $[a, b]$ to be the union of the sets G_n for $n = 1, 2, \ldots$ will be at most countable. Assume, on the contrary, that G_n contains infinitely many points $t_1, t_2, \ldots$ so that

$$\left| f(t_k + 0) - f(t_k - 0) \right| \geqslant \frac{1}{n} \quad (k = 1, 2, \ldots). \tag{5.21}$$

Since $t_k \in [a, b]$ for all $k = 1, 2, \ldots$, we may assume, passing to a subsequence of $\{t_k\}$ if it is necessary, that

$$\lim_{k\to\infty} t_k = t_0 \quad \text{for some } t_0 \in [a, b]. \tag{5.22}$$

Note that $t_0 \in \mathbf{T}$ since $t_k \in \mathbf{T}$ $(k = 1, 2, \ldots)$ and $\mathbf{T}$ is closed. By (5.22) there exists a subsequence that tends to t_0 from above or a subsequence that tends to t_0 from below, and in any case the limit of $f(t)$ as $t \to t_0$ has to exist (finite) according to regularity. But this contradicts (5.21).

Applying Theorem 5.8 we get that every regulated function $f : [a, b] \to \mathbf{R}$, in particular, every monotonic function on $[a, b]$, is Riemann Δ-integrable from a to b.

Note that the statements of all theorems of Section 3 on Riemann Δ-integrability can also be obtained as immediate consequences of Theorem 5.8.

Finally, let us now formulate analogues of Theorems 5.7 and 5.8 for the nabla integral.

Theorem 5.9. *Let $(a, b]$ be a half-closed bounded interval in $\mathbf{T}$, and let f be a bounded real-valued function on $(a, b]$. If f is Riemann ∇-integrable from a to b, then f is Lebesgue ∇-integrable on $(a, b]$, and $R\int_a^b f(t)\,\nabla t = L\int_{(a,b]} f(t)\,\nabla t$.*

Theorem 5.10. *Let f be a bounded function defined on the half-closed bounded interval $(a, b]$ of $\mathbf{T}$. Then f is Riemann ∇-integrable from a to b if and only if the set of all left-dense points of $(a, b]$ at which f is discontinuous is a set of ∇-measure zero.*

Appendix A. Mean value results on time scales

For the theory of calculus on time scales we refer to the original work by Hilger [16], to the paper by Aulbach and Hilger [6], and to the recently appeared works [1,2,4,5,8,9, 12,17,19]. In this section we give basic definitions concerning the time scales and present some mean value theorems for derivatives on time scales.

A *time scale* $\mathbf{T}$ (which is a special case of a *measure chain*) is an arbitrary nonempty closed subset of the real numbers $\mathbf{R}$, therefore it is a complete metric space with the metric $d(t, s) = |t - s|$. For $t \in \mathbf{T}$ we define the *forward jump operator* $\sigma : \mathbf{T} \to \mathbf{T}$ by $\sigma(t) = \inf\{s \in \mathbf{T}:\ s > t\}$, while the *backward jump operator* $\rho : \mathbf{T} \to \mathbf{T}$ is defined by $\rho(t) = \sup\{s \in \mathbf{T}:\ s < t\}$. In this definition we put in addition $\sigma(\max \mathbf{T}) = \max \mathbf{T}$ if there exists a finite $\max \mathbf{T}$, and $\rho(\min \mathbf{T}) = \min \mathbf{T}$ if there exists a finite $\min \mathbf{T}$. Of course both $\sigma(t)$ and $\rho(t)$ are in $\mathbf{T}$ when $t \in \mathbf{T}$. This is because of our assumption that $\mathbf{T}$ is a closed subset of $\mathbf{R}$.

If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$ we say that t is *left-scattered*. Also, if $t < \max \mathbf{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \min \mathbf{T}$ and $\rho(t) = t$, then t is called *left-dense*.

We introduce the sets $\mathbf{T}^k$ and $\mathbf{T}_k$ which are derived from the time scale $\mathbf{T}$ as follows. If $\mathbf{T}$ has a left-scattered maximum t_1, then $\mathbf{T}^k = \mathbf{T} - \{t_1\}$, otherwise $\mathbf{T}^k = \mathbf{T}$. If $\mathbf{T}$ has a right-scattered minimum t_2, then $\mathbf{T}_k = \mathbf{T} - \{t_2\}$, otherwise $\mathbf{T}_k = \mathbf{T}$.

For $a, b \in \mathbf{T}$ with $a \leqslant b$ we define the closed interval $[a, b]$ in $\mathbf{T}$ by $[a, b] = \{t \in \mathbf{T}:\ a \leqslant t \leqslant b\}$. Open intervals and half-open intervals etc. are defined accordingly.

If $f : \mathbf{T} \to \mathbf{R}$ is a function and $t \in \mathbf{T}^k$, then the *delta* (or *Hilger*) *derivative* of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\varepsilon > 0$ there is a neighborhood U (in $\mathbf{T}$) of t such that

$$\left|f\big(\sigma(t)\big) - f(s) - f^{\Delta}(t)\big[\sigma(t) - s\big]\right| \leqslant \varepsilon\left|\sigma(t) - s\right| \quad \text{for all } s \in U.$$

If $t \in \mathbf{T}_k$, then we define the *nabla derivative* of f at the point t to be the number $f^{\nabla}(t)$ (provided it exists) with the property that for each $\varepsilon > 0$ there is a neighborhood U (in $\mathbf{T}$) of t such that

$$\left|f\big(\rho(t)\big) - f(s) - f^{\nabla}(t)\big[\rho(t) - s\big]\right| \leqslant \varepsilon\left|\rho(t) - s\right| \quad \text{for all } s \in U.$$

Note that the delta and nabla derivatives are particular cases of the alpha derivative introduced by Ahlbrandt et al. in [2], namely with $\alpha = \sigma$ and $\alpha = \rho$, respectively.

If $\mathbf{T} = \mathbf{R}$ then f is delta differentiable (nabla differentiable) at t if f is differentiable in the ordinary sense at t. In this case we then have $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$. If $\mathbf{T} = \mathbf{Z}$, then f is delta differentiable (nabla differentiable) at t and we have

$$f^{\Delta}(t) = f(t+1) - f(t) = \Delta f(t), \qquad f^{\nabla}(t) = f(t) - f(t-1) = \nabla f(t),$$

where Δ is the usual *forward difference operator* and ∇ is the usual *backward difference operator* defined by the last equations above.

Theorem A.1. *Let f be a continuous function on $[a, b]$ that is Δ-differentiable on $[a, b)$ (the differentiability at a is understood as right-sided) and satisfies $f(a) = f(b)$. Then there exist $\xi, \tau \in [a, b)$ such that $f^{\Delta}(\tau) \leqslant 0 \leqslant f^{\Delta}(\xi)$.*

Proof. Since the function f is continuous on the compact set $[a, b]$, f assumes its minimum m and its maximum M. Therefore there exist $\xi, \tau \in [a, b]$ such that $m = f(\xi)$ and $M = f(\tau)$. Since $f(a) = f(b)$, we may assume that $\xi, \tau \in [a, b)$. Clearly we have $f^{\Delta}(\tau) \leqslant 0$ and $f^{\Delta}(\xi) \geqslant 0$. $\square$

Theorem A.2 (Mean value theorem). *Let f be a continuous function on $[a, b]$ which is Δ-differentiable on $[a, b)$. Then there exist $\xi, \tau \in [a, b)$ such that*

$$f^{\Delta}(\tau) \leqslant \frac{f(b) - f(a)}{b - a} \leqslant f^{\Delta}(\xi).$$

The proof is obtained by applying Theorem A.1 to the function

$$\varphi(t) = f(t) - f(a) - \frac{f(b) - f(a)}{b - a}(t - a).$$

Corollary A.3. *Let f be a continuous function on $[a, b]$ that is Δ-differentiable on $[a, b)$. If $f^{\Delta}(t) = 0$ for all $t \in [a, b)$, then f is a constant function on $[a, b]$.*

Corollary A.4. *Let f be a continuous function on $[a, b]$ that is Δ-differentiable on $[a, b)$. Then f is increasing, decreasing, nondecreasing, and nonincreasing on $[a, b]$ if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geqslant 0$, and $f^{\Delta}(t) \leqslant 0$ for all $t \in [a, b)$, respectively.*

The following analogue for ∇-derivative of Theorem A.2 can be proved in a similar way.

Theorem A.5. *Let f be a continuous function on $[a, b]$ that is ∇-differentiable on $(a, b]$. Then there exist $\xi, \tau \in (a, b]$ such that*

$$f^{\nabla}(\tau) \leqslant \frac{f(b) - f(a)}{b - a} \leqslant f^{\nabla}(\xi).$$

Let us now establish a generalization of Theorem A.2 for so-called Δ-predifferentiable functions.

Definition A.6 (see [16, p. 30]). A function $f:\mathbf{T}\to\mathbf{R}$ is called *Δ-predifferentiable* (with region of Δ-differentiation D), provided that the following conditions hold:

(i) f is continuous on $\mathbf{T}$;
(ii) $D\subset\mathbf{T}^k$, $\mathbf{T}^k-D$ is countable and contains no right-scattered elements of $\mathbf{T}$;
(iii) f is Δ-differentiable at each $t\in D$.

We first formulate the following version of mean value theorem proved in Hilger's paper [16] (see also [9, Chap. 1]).

Theorem A.7. *Let f and g be real-valued functions defined on $\mathbf{T}$. Suppose both f and g are Δ-predifferentiable with region of Δ-differentiation D. Then $|f^{\Delta}(t)|\leqslant g^{\Delta}(t)$ for all $t\in D$ implies $|f(r)-f(s)|\leqslant g(r)-g(s)$ for all $r,s\in\mathbf{T},\ r\geqslant s$.*

Taking in Theorem A.7, in particular, $f=0$, we obtain:

Corollary A.8. *Let $g:\mathbf{T}\to\mathbf{R}$ be a Δ-predifferentiable function with region of Δ-differentiation D. If $g^{\Delta}(t)\geqslant 0$ for all $t\in D$, then g is nondecreasing on $\mathbf{T}$.*

Theorem A.9. *Let f be a continuous function on $[a,b]\subset\mathbf{T}$ that is Δ-predifferentiable on $[a,b)$ with region of Δ-differentiation $D\subset[a,b)$. Suppose $f(a)=f(b)$. Then there exist $\xi,\tau\in D$ such that $f^{\Delta}(\tau)\leqslant 0\leqslant f^{\Delta}(\xi)$.*

Proof. If f is a constant function, then $f^{\Delta}(t)=0$ for all $t\in[a,b)$ and, therefore, the theorem holds in this case. Now suppose that *f is* not constant. To prove that there exists $\tau\in D$ such that $f^{\Delta}(\tau)\leqslant 0$, we suppose the contrary: let $f^{\Delta}(t)>0$ for all $t\in D$. Applying Corollary A.8 to the function $f:[a,b]\to\mathbf{R}$ we get that f is nondecreasing on $[a,b]$. But this gives a contradiction, since $f(a)=f(b)$ and f is nonconstant. Therefore the desired point $\tau\in D$ exists. Similarly, considering the function $-f$, we can prove that there exists $\xi\in D$ such that $f^{\Delta}(\xi)\geqslant 0$. □

Now using Theorem A.9, we can prove as in the proof of Theorem A.2 that the following generalization of Theorem A.2 is true.

Theorem A.10. *Let f be a continuous function on $[a,b]\subset\mathbf{T}$ that is Δ-predifferentiable on $[a,b)$ with region of Δ-differentiation $D\subset[a,b)$. Then there exist $\xi,\tau\in D$ such that $(b-a)f^{\Delta}(\tau)\leqslant f(b)-f(a)\leqslant(b-a)f^{\Delta}(\xi)$.*

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